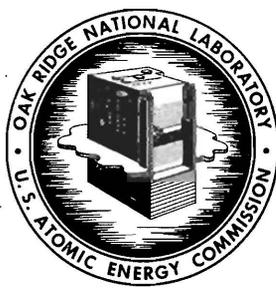


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SOME SIMPLIFIED EXPRESSIONS FOR THE $3-j$ SYMBOLS

C. D. Zerby and R. R. Coveyou

ABSTRACT

A general formalism is developed for obtaining simplified expressions for the $3-j$ symbol using the symmetry properties of the symbol discovered by Regge. Some new formulas are deduced from the formalism for cases in which Racah's series for the symbol is summable.

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SOME SIMPLIFIED EXPRESSIONS FOR THE 3-j SYMBOLS

Values of the 3-j symbol, represented by the notation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad (1)$$

can be calculated using the formula given by Racah:¹

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{12}+A_{33}} \left[\frac{A_{11}! A_{12}! A_{13}! A_{21}! A_{22}! A_{23}! A_{31}! A_{32}! A_{33}!}{(J+1)!} \right]^{1/2} \\ \times \sum_p \frac{(-1)^p}{p!(A_{13}-p)!(A_{21}-p)!(A_{32}-p)!(A_{12}-A_{21}+p)!(A_{11}-A_{32}+p)!} \quad (2)$$

where each A_{ij} is a positive integer or zero and the sum of each column or row is the integer $J = j_1 + j_2 + j_3$.

It is well known that the number of terms in the sum in Eq. 2 is equal to one more than the smallest A_{ij} .¹ Therefore, when any $A_{ij} = 0$ the sum in Eq. 2 reduces to a single term, and a simplified expression for the 3-j symbol results.

The only other simplified expression for the 3-j symbol was given by Racah¹ for $m_1 = m_2 = m_3 = 0$, in which case the series given in Eq. 2 is summable.

The purpose of this paper is to show that with the aid of the symmetry properties of the 3-j symbol discovered by Regge,² a single expression can be written for the case where any $A_{ij} = 0$. In addition, it will be shown that



the case reported by Racah in which the series is summable is only a special example of a more general case.

First, it is important to note that the symmetry properties of the $3-j$ symbol was displayed very nicely by using the square array which was introduced by Regge² and is shown in Eqs. 1 and 2. The square array is invariant to reflections about the diagonals and is multiplied by $(-1)^J$ upon exchanging two adjacent rows or columns.

Equation 2 can be written in a more suitable form for present purposes if the sum is expressed in terms of A_{11} , A_{22} , A_{23} , A_{32} , and A_{33} using the relations

$$\begin{aligned}
 A_{12} &= -A_{11} + A_{33} + A_{23} \\
 A_{13} &= -A_{11} + A_{22} + A_{32} \\
 A_{21} &= -A_{11} + A_{33} + A_{32} \\
 A_{31} &= -A_{11} + A_{22} + A_{23}
 \end{aligned}
 \tag{3}$$

By changing the summation parameter to $k = A_{11} - A_{32} + p$ in Eq. 2, we obtain

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}! A_{13}! A_{21}! A_{31}!}{A_{11}! A_{22}! A_{33}! A_{23}! A_{32}! (J+1)!} \right]^{1/2}$$

$$\times \begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix}
 \tag{4}$$

where

$$\begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} = \sum_k \frac{(-1)^k A_{11}! A_{22}! A_{33}! A_{23}! A_{32}!}{k! (A_{11}-k)! (A_{22}-k)! (A_{33}-k)! (A_{23}-A_{11}+k)! (A_{32}-A_{11}+k)!} \quad (4a)$$

The sum indicated in Eq. 4a is always an integer or zero since each term in the sum is an integer. When A_{11} is equal to zero, the summation index, k , can only assume the value zero and the sum is equal to unity. Hence, we have the first simplified formula

$$\begin{Bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}! A_{13}! A_{21}! A_{31}!}{A_{22}! A_{33}! A_{23}! A_{32}! (J+1)!} \right]^{1/2} \quad (5)$$

which applies to the case where any element $A_{ij} = 0$, since that element can always be transformed to the A_{11} position using the symmetry properties of the 3-j symbol. From Eq. 5 we can easily obtain the well-known explicit formulas³ for the cases $j_k = j_\ell + j_m$ and $m_k = \pm j_k$.

Next we consider the cases where the series given in Eq. 2 is summable. We use the expression derived by Racah¹

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{bmatrix} = \frac{(-1)^{\frac{j_1+j_2+j_3}{2}}}{(j_1+j_2+j_3+1)^{1/2}} \frac{S(j_1+j_2+j_3)}{S(-j_1+j_2+j_3) S(j_1-j_2+j_3) S(j_1+j_2-j_3)} \quad (6)$$

where $S(x) = (x/2)! (x!)^{-1/2}$. Equation 6 is valid only when $J = j_1 + j_2 + j_3$ is even; otherwise the symbol is equal to zero.

Substituting $A_{2k} = A_{3k} = j_k$ ($k = 1, 2, 3$) in Eq. 6 and using the relations given in Eq. 3, we find

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{33} \\ A_{21} & A_{22} & A_{33} \end{bmatrix} = \frac{(-1)^{J/2}}{(J+1)^{1/2}} \frac{S(J)}{S(A_{11}) S(A_{12}) S(A_{13})} \quad (7)$$

which is valid for J even and for all allowed values of A_{ij} . For J odd the symbol is equal to zero. Using the symmetry properties of the 3-j symbol, we conclude that the series is summable for the cases where any two rows or two columns are identical.

From Eq. 7 we obtain Eq. 6 and the new explicit expressions

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \pm[2j_1-j_2-j_3] & \pm[2j_2-j_1-j_3] & \pm[2j_3-j_2-j_1] \end{pmatrix} = \frac{(-1)^{\frac{j_1+j_2+j_3}{2}}}{(j_1+j_2+j_3+1)^{1/2}} \frac{S(j_1+j_2+j_3)}{S(3j_1-j_2-j_3) S(3j_2-j_1-j_3) S(3j_3-j_1-j_2)} \quad (8)$$

and

$$\begin{pmatrix} j_1 & j_2 & j_2 \\ -2m_2 & m_2 & m_2 \end{pmatrix} = \frac{(-1)^{\frac{j_1}{2} + j_2}}{(j_1+2j_2+1)^{1/2}} \frac{S(j_1+2j_2)}{S(-j_1+2j_2) S(j_1-2m_2) S(j_1+2m_2)} \quad (9)$$

where $j_1+j_2+j_3$ must be even in Eq. 8 and j_1+2j_2 must be even in Eq. 9. Equation 9 is also invariant to the ordering of the columns in the $3-j$ symbol.

It is also of some interest to display a few formulas applicable to the cases where the smallest A_{ij} is greater than zero. To obtain these expressions we use the recursion relation⁴

$$\begin{aligned}
 [(J+1)(J-2j_1)]^{1/2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} &= [(j_2+m_2)(j_3-m_3)]^{1/2} \begin{pmatrix} j_1 & j_2-1/2 & j_3-1/2 \\ m_1 & m_2-1/2 & m_3+1/2 \end{pmatrix} \\
 &- [(j_2-m_2)(j_3+m_3)]^{1/2} \begin{pmatrix} j_1 & j_2-1/2 & j_3-1/2 \\ m_1 & m_2+1/2 & m_3-1/2 \end{pmatrix} \quad (10)
 \end{aligned}$$

Making use of Eqs. 1 and 4 in Eq. 10, we find the following recursion relationship

$$\begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} = A_{32} A_{23} \begin{Bmatrix} A_{11}-1 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23}-1 \\ A_{31} & A_{32}-1 & A_{33} \end{Bmatrix} - A_{22} A_{33} \begin{Bmatrix} A_{11}-1 & A_{12} & A_{13} \\ A_{21} & A_{22}-1 & A_{23} \\ A_{31} & A_{32} & A_{33}-1 \end{Bmatrix} \quad (11)$$

Remembering that the sum given in Eq. 4a equals unity when $A_{11} = 0$, we can then deduce from Eqs. 11 and 4 that

$$\begin{Bmatrix} 1 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}! A_{13}! A_{21}! A_{31}!}{A_{22}! A_{33}! A_{23}! A_{32}! (J+1)!} \right]^{1/2} \times [A_{23} A_{32} - A_{22} A_{33}] \quad (12)$$

and

$$\begin{bmatrix} 2 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}! A_{13}! A_{21}! A_{31}!}{2 A_{22}! A_{33}! A_{23}! A_{32}! (J+1)!} \right]^{1/2}$$

$$\left\{ A_{32}A_{23}[(A_{32}-1)(A_{23}-1) - A_{22}A_{33}] - A_{22}A_{33}[A_{32}A_{23} - (A_{22}-1)(A_{33}-1)] \right\} \quad (13)$$

etc. From Eq. 12 it is easy to obtain explicit formulas for $j_k = j_\ell + j_m - 1$ and $m_k = \pm(j_k - 1)$. From Eq. 13 formulas for $j_k = j_\ell + j_m - 2$ and $m_k = \pm(j_k - 2)$ can be obtained.

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