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SOME SIMPLIFIED EXPRESSIONS FOR THE 3-j SYMBOLS

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ABSTRACT

A general formalism is developed for obtaining simplified expressions for the 3-j symbol using the symmetry properties of the symbol discovered by Regge. Some new formulas are deduced from the formalism for cases in which Racah's series for the symbol is summable.

SOME SIMPLIFIED EXPRESSIONS FOR THE 3-j SYMBOLS

Values of the 3-j symbol, represented by the notation

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{1}$$

can be calculated using the formula given by Racah:[1]

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{12}+A_{33}} \left[\frac{A_{11}!\ A_{12}!\ A_{13}!\ A_{21}!\ A_{22}!\ A_{23}!\ A_{31}!\ A_{32}!\ A_{33}!}{(J+1)!} \right]^{1/2}$$

$$\times \sum_p \frac{(-1)^p}{p!(A_{13}-p)!\ (A_{21}-p)!\ (A_{32}-p)!\ (A_{12}-A_{21}+p)!\ (A_{11}-A_{32}+p)!} \tag{2}$$

where each A_{ij} is a positive integer or zero and the sum of each column or row is the integer $J = j_1 + j_2 + j_3$.

It is well known that the number of terms in the sum in Eq. 2 is equal to one more than the smallest A_{ij}.[1] Therefore, when any $A_{ij} = 0$ the sum in Eq. 2 reduces to a single term, and a simplified expression for the 3-j symbol results.

The only other simplified expression for the 3-j symbol was given by Racah[1] for $m_1 = m_2 = m_3 = 0$, in which case the series given in Eq. 2 is summable.

The purpose of this paper is to show that with the aid of the symmetry properties of the 3-j symbol discovered by Regge,[2] a single expression can be written for the case where any $A_{ij} = 0$. In addition, it will be shown that

the case reported by Racah in which the series is summable is only a special example of a more general case.

First, it is important to note that the symmetry properties of the 3-j symbol was displayed very nicely by using the square array which was introduced by Regge[2] and is shown in Eqs. 1 and 2. The square array is invariant to reflections about the diagonals and is multiplied by $(-1)^J$ upon exchanging two adjacent rows or columns.

Equation 2 can be written in a more suitable form for present purposes if the sum is expressed in terms of A_{11}, A_{22}, A_{23}, A_{32}, and A_{33} using the relations

$$\begin{aligned} A_{12} &= -A_{11} + A_{33} + A_{23} \\ A_{13} &= -A_{11} + A_{22} + A_{32} \\ A_{21} &= -A_{11} + A_{33} + A_{32} \\ A_{31} &= -A_{11} + A_{22} + A_{23} \end{aligned} \qquad (3)$$

By changing the summation parameter to $k = A_{11} - A_{32} + p$ in Eq. 2, we obtain

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}!\, A_{13}!\, A_{21}!\, A_{31}!}{A_{11}!\, A_{22}!\, A_{33}!\, A_{23}!\, A_{32}!\, (J+1)!} \right]^{1/2}$$

$$\times \begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} \qquad (4)$$

where

$$\begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} = \sum_k \frac{(-1)^k A_{11}! \quad A_{22}! \quad A_{33}! \quad A_{23}! \quad A_{32}!}{k!\ (A_{11}-k)!\ (A_{22}-k)!\ (A_{33}-k)!\ (A_{23}-A_{11}+k)!\ (A_{32}-A_{11}+k)!} \tag{4a}$$

The sum indicated in Eq. 4a is always an integer or zero since each term in the sum is an integer. When A_{11} is equal to zero, the summation index, k, can only assume the value zero and the sum is equal to unity. Hence, we have the first simplified formula

$$\begin{bmatrix} 0 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}!\ A_{13}!\ A_{21}!\ A_{31}!}{A_{22}!\ A_{33}!\ A_{23}!\ A_{32}!\ (J+1)!} \right]^{1/2} \tag{5}$$

which applies to the case where any element $A_{ij} = 0$, since that element can always be transformed to the A_{11} position using the symmetry properties of the 3-j symbol. From Eq. 5 we can easily obtain the well-known explicit formulas[3] for the cases $j_k = j_\ell + j_m$ and $m_k = \pm j_k$.

Next we consider the cases where the series given in Eq. 2 is summable. We use the expression derived by Racah[1]

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ 0 & 0 & 0 \end{pmatrix} = \begin{bmatrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{bmatrix} =$$

$$= \frac{(-1)^{\frac{j_1+j_2+j_3}{2}}}{(j_1+j_2+j_3+1)^{1/2}} \quad \frac{S(j_1+j_2+j_3)}{S(-j_1+j_2+j_3)\ S(j_1-j_2+j_3)\ S(j_1+j_2-j_3)} \tag{6}$$

where $S(x) = (x/2)!\,(x!)^{-1/2}$. Equation 6 is valid only when $J = j_1 + j_2 + j_3$ is even; otherwise the symbol is equal to zero.

Substituting $A_{2k} = A_{3k} = j_k$ $(k = 1,2,3)$ in Eq. 6 and using the relations given in Eq. 3, we find

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{33} \\ A_{21} & A_{22} & A_{33} \end{bmatrix} = \frac{(-1)^{J/2}}{(J+1)^{1/2}} \quad \frac{S(J)}{S(A_{11})\; S(A_{12})\; S(A_{13})} \tag{7}$$

which is valid for J even and for all allowed values of A_{ij}. For J odd the symbol is equal to zero. Using the symmetry properties of the 3-j symbol, we conclude that the series is summable for the cases where any two rows or two columns are identical.

From Eq. 7 we obtain Eq. 6 and the new explicit expressions

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ \pm[2j_1-j_2-j_3] & \pm[2j_2-j_1-j_3] & \pm[2j_3-j_2-j_1] \end{pmatrix} =$$

$$= \frac{(-1)^{\frac{j_1+j_2+j_3}{2}}}{(j_1+j_2+j_3+1)^{1/2}} \quad \frac{S(j_1+j_2+j_3)}{S(3j_1-j_2-j_3)\; S(3j_2-j_1-j_3)\; S(3j_3-j_1-j_2)} \tag{8}$$

and

$$\begin{pmatrix} j_1 & j_2 & j_2 \\ -2m_2 & m_2 & m_2 \end{pmatrix} = \frac{(-1)^{\frac{j_1}{2}+j_2}}{(j_1+2j_2+1)^{1/2}} \quad \frac{S(j_1+2j_2)}{S(-j_1+2j_2)\; S(j_1-2m_2)\; S(j_1+2m_2)} \tag{9}$$

where $j_1+j_2+j_3$ must be even in Eq. 8 and j_1+2j_2 must be even in Eq. 9. Equation 9 is also invariant to the ordering of the columns in the 3-j symbol.

It is also of some interest to display a few formulas applicable to the cases where the smallest A_{ij} is greater than zero. To obtain these expressions we use the recursion relation[4]

$$[(J+1)(J-2j_1)]^{1/2}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = [(j_2+m_2)(j_3-m_3)]^{1/2}\begin{pmatrix} j_1 & j_2-1/2 & j_3-1/2 \\ m_1 & m_2-1/2 & m_3+1/2 \end{pmatrix}$$

$$- [(j_2-m_2)(j_3+m_3)]^{1/2}\begin{pmatrix} j_1 & j_2-1/2 & j_3-1/2 \\ m_1 & m_2+1/2 & m_3-1/2 \end{pmatrix} \qquad (10)$$

Making use of Eqs. 1 and 4 in Eq. 10, we find the following recursion relationship

$$\begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix} = A_{32}\, A_{23} \begin{Bmatrix} A_{11}-1 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23}-1 \\ A_{31} & A_{32}-1 & A_{33} \end{Bmatrix} - A_{22}\, A_{33} \begin{Bmatrix} A_{11}-1 & A_{12} & A_{13} \\ A_{21} & A_{22}-1 & A_{23} \\ A_{31} & A_{32} & A_{33}-1 \end{Bmatrix} \qquad (11)$$

Remembering that the sum given in Eq. 4a equals unity when $A_{11} = 0$, we can then deduce from Eqs. 11 and 4 that

$$\begin{bmatrix} 1 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}!\, A_{13}!\, A_{21}!\, A_{31}!}{A_{22}!\, A_{33}!\, A_{23}!\, A_{32}!\, (J+1)!} \right]^{1/2}$$

$$\times\ [A_{23}\, A_{32} - A_{22}\, A_{33}] \qquad (12)$$

and

$$\begin{bmatrix} 2 & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = (-1)^{A_{23}+A_{32}} \left[\frac{A_{12}!\ A_{13}!\ A_{21}!\ A_{31}!}{2\ A_{22}!\ A_{33}!\ A_{23}!\ A_{32}!\ (J+1)!} \right]^{1/2}$$

$$\left\{ A_{32}A_{23}[(A_{32}-1)(A_{23}-1) - A_{22}A_{33}] - A_{22}A_{33}[A_{32}A_{23} - (A_{22}-1)(A_{33}-1)] \right\} \qquad (13)$$

etc. From Eq. 12 it is easy to obtain explicit formulas for $j_k = j_\ell + j_m - 1$ and $m_k = \pm(j_k - 1)$. From Eq. 13 formulas for $j_k = j_\ell + j_m - 2$ and $m_k = \pm(j_k - 2)$ can be obtained.

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