RELATIVISTIC GENERALIZATION OF THE POST-PRIOR EQUIVALENCE FOR REACTION OF COMPOSITE PARTICLES

CHEUK-YIN WONG[1] and HORACE W. CRATER[2]

[1]*Physics Division, Oak Ridge National Laboratory, Oak Ridge, Tennessee 37831, U.S.A.*
[2]*Department of Physics, University of Tennessee Space Institute, Tullahoma, Tennessee 37388, U.S.A.*
E-mail: wongc@ornl.gov

In the non-relativistic description of the reaction of composite particles, the reaction matrix is independent of the choice of post or prior forms for the interaction. We generalize this post-prior equivalence to the relativistic reaction of composite particles by using Dirac's constraint dynamics to describe the bound states and the reaction process.

1 Introduction

In the non-relativistic description of composite particle reactions, it is well known that the reaction matrix element is independent of the choice of post or prior forms for the interaction[1]. Specifically, for the reaction of composite particles A, B, C and D with constituents 1, 2, 3, and 4,

$$A(12) + B(34) \rightarrow C(14) + D(32), \qquad (1)$$
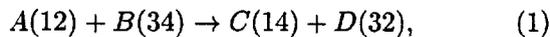
the prior reaction matrix element between the initial state $|i\rangle$ and the final state $|f\rangle$, $\langle f|V_{13}+V_{14}+V_{23}+V_{24}|i\rangle$, is equal to the post reaction matrix element, $\langle f|V_{13} + V_{12} + V_{43} + V_{42}|i\rangle$, where V_{ij} is the interaction between constituents i and j. This equality follows if and only if the interaction which induces the reaction is the same as the interaction which generates the composite bound states, and the reaction matrix element is evaluated using these bound state wave functions. The post-prior equivalence guarantees the uniqueness of the reaction cross section in the Born approximation.

Recently there has been much interest in studying the reaction of composite particles at relativistic energies in nuclear and particle physics (see Refs. [2,3] and references cited therein). It is clearly of interest to derive a relativistic generalization of the post-prior theorem. We shall briefly summarize the proof of the post-prior equivalence in relativistic reactions of composite particles using Dirac's constraint dynamics[4]. A more detailed description can be found in Ref. [3].

2 Relativistic 2-Body Bound State Problem

Before we examine the reaction of composite two-body systems, it is important to discuss the structure of the bound states. Much progress has been made in the study of the relativistic two-body bound state problem using Dirac's relativistic constraint dynamics[4,5,6,7]. This constraint formalism represents a simplified, yet complete, resummation of the Bethe-Salpeter equation. It is similar to many other three-dimensional quasipotential truncations, but has a number of advantages over many of those approaches. We briefly review here the results for bound states of spinless particles. One assumes a generalized mass shell constraint for each particle

$$\mathcal{H}_i|\psi\rangle = 0 \qquad \text{for} \qquad i = 1, 2 \qquad (2)$$

where

$$\mathcal{H}_i = p_i^2 - m_i^2 - \Phi_i, \qquad (3)$$

and Φ_1 and Φ_2 are two-body interactions dependent on coordinate x_{12}. One constructs the total Hamiltonian $\mathcal{H}$

$$\mathcal{H} = \lambda_1 \mathcal{H}_1 + \lambda_2 \mathcal{H}_2, \qquad (4)$$

where $\{\lambda_i\}$ are Lagrange multipliers. In order that the constraints be compatible, we must have

$$[\mathcal{H}_1, \mathcal{H}_2]|\psi\rangle = 0. \qquad (5)$$

The simplest way to satisfy the above condition is to take

$$\Phi_1 = \Phi_2 = \Phi(x_\perp), \qquad (6)$$

where

$$x_{\nu\perp} = x_{12}^\mu(\eta_{\mu\nu} - P_\mu P_\nu / P^2), \qquad (7)$$

and P is the total momentum $P = p_1 + p_2$.

The equation of motion, $\mathcal{H}|\psi\rangle = 0$, describes both the center-of-mass and the relative motion. To separate the center-of-mass and the relative motion, we introduce the relative momentum q

$$p_1 = \frac{p_1 \cdot P}{P^2} P + q, \qquad (8)$$

$$p_2 = \frac{p_2 \cdot P}{P^2} P - q. \qquad (9)$$

We then obtain $\mathcal{H}$ in terms of P and q:

$$\mathcal{H}|\psi\rangle = (\lambda_1 + \lambda_2)[q^2 - \Phi(x_\perp) + b^2(P^2; m_1^2, m_2^2)]|\psi\rangle = 0,$$

where

$$b^2(P^2, m_1^2, m_2^2) = \frac{1}{4P^2}[P^4 - 2P^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2].$$

The equation of motion, $\mathcal{H}|\psi\rangle = 0$, can be solved by introducing the bound state mass M to separate it into the following two equations for center-of-mass motion and relative motion

$$\{P^2 - M^2\}|\psi\rangle = 0, \tag{10}$$

$$(\lambda_1 + \lambda_2)\{q^2 - \Phi(x_\perp) + b^2(M^2, m_1^2, m_2^2)\}|\psi\rangle = 0. \tag{11}$$

The equation for relative motion is independent of λ_i. We go to the center-of-momentum system where $q = q_\perp = (0, \mathbf{q})$ and $x_\perp = (0, \mathbf{r})$. It is convenient to choose $\lambda_i = 1/2m_i$ so that the equation for relative motion, Eq. (11), matches the Schrödinger equation term by term:

$$\left\{\frac{\mathbf{q}^2}{2\mu} + \frac{\Phi(\mathbf{r})}{2\mu}\right\}|\psi\rangle = \frac{b^2}{2\mu}|\psi\rangle = E|\psi\rangle, \tag{12}$$

where $\mu = m_1 m_2/(m_1 + m_2)$. From the eigenvalue of this Schrödinger equation and $b^2(M^2, m_1^2, m_2^2) = 2\mu E$, we can obtain the bound state mass M given by

$$M = \sqrt{2\mu E + m_1^2} + \sqrt{2\mu E + m_2^2}. \tag{13}$$

3 Scattering of Two Composite Particles

For a system of two composite particles with a total of 4 constituents, we can similarly specify a generalized mass shell constraint for each constituent:

$$\mathcal{H}_i|\psi\rangle = 0 \qquad \text{for} \qquad i = 1, .., 4 \tag{14}$$

where [8]

$$\mathcal{H}_i = p_i^2 - m_i^2 - \sum_{j \neq i}^{4} \Phi_{ij}. \tag{15}$$

We construct the total relativistic Hamiltonian as

$$\mathcal{H} = \sum_{i=1}^{4} \lambda_i \mathcal{H}_i, \tag{16}$$

and choose $\lambda_i = 1/2m_i$. The total Hamiltonian is then

$$\mathcal{H} = \sum_{i=1}^{4} \frac{p_i^2 - m_i^2}{2m_i} - \sum_{i=1}^{4}\sum_{j \neq i}^{4} V_{ij}, \tag{17}$$

where $\mu_{ij} = m_i m_j/(m_i + m_j)$ and $V_{ij} = \Phi_{ij}/2\mu_{ij}$.

The quadratic form of the momentum operators p_i^2 in this 4-body Hamiltonian simplifies the separation the center-of-mass momentum and the relative momentum for any pair of particles. One can therefore use the two-body Hamiltonians to generate basis states. Specifically,

we solve for the bound states ψ_{ij} of mass M_{ij} for particles i and j interacting with the interaction V_{ij},

$$\mathcal{H}_{ij}|\psi_{ij}\rangle = \left\{\frac{p_i^2 - m_i^2}{2m_i} + \frac{p_j^2 - m_j^2}{2m_j} - V_{ij}\right\}|\psi_{ij}\rangle = 0. \tag{18}$$

To study the relativistic constituent-interchange process of Eq. (1) as a generalization of the non-relativistic case investigated by Barnes and Swanson [9], we can partition the total Hamiltonian into an unperturbed part $\mathcal{H}_0$ and a residual interaction V_I in two different ways. In the "prior" form, this partition is

$$\mathcal{H} = \mathcal{H}_{12} + \mathcal{H}_{34} - V_{13} - V_{14} - V_{23} - V_{24}, \tag{19}$$

$$\mathcal{H}_0(\text{prior}) = \mathcal{H}_{12} + \mathcal{H}_{34}, \tag{20}$$

$$V_I(\text{prior}) = -V_{13} - V_{14} - V_{23} - V_{24}. \tag{21}$$

The corresponding "prior" reaction matrix element is

$$2\pi\delta^{(4)}(P_A + P_B - P_C - P_D)h_{fi}(\text{prior}) = -\langle\psi_{14}\psi_{23}|V_{13} + V_{14} + V_{23} + V_{24}|\psi_{12}\psi_{34}\rangle. \tag{22}$$

If we represent the interaction V_{ij} n diagrammatic form [9] by a curly line, the four terms in the above matrix element are represented by the four diagrams in Fig. 1. The interaction takes place before the interchange of constituents.
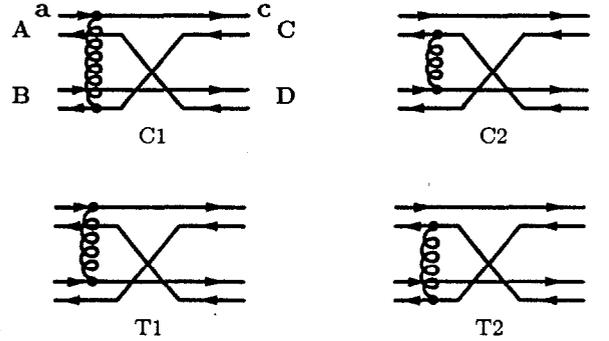

Fig. 1. 'Prior' diagrams for the reaction A+B → C+D.

On the other hand, in the "post" form of partitioning the total Hamiltonian, we have

$$\mathcal{H} = \mathcal{H}_{14} + \mathcal{H}_{32} - V_{13} - V_{12} - V_{43} - V_{42}, \tag{23}$$

$$\mathcal{H}_0(\text{post}) = \mathcal{H}_{14} + \mathcal{H}_{32}, \tag{24}$$

$$V_I(\text{post}) = -V_{13} - V_{12} - V_{43} - V_{42}. \tag{25}$$

The "post" reaction matrix element is

$$2\pi\delta^4(P_A + P_B - P_C - P_D)h_{fi}(\text{post}) = -\langle\psi_{14}\psi_{23}|V_{12} + V_{13} + V_{42} + V_{43}|\psi_{12}\psi_{34}\rangle. \tag{26}$$

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The four terms in the post matrix element are represented by the four diagrams of Fig. 2. The interaction takes place after the interchange of constituents.
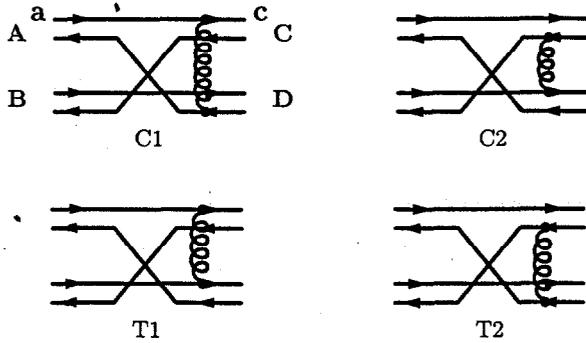

Fig. 2. 'Post' diagrams for the reaction $A+B \rightarrow C+D$.

If we start with the prior expression for the matrix element, since $\mathcal{H}_{12}|\psi_{12}\rangle = 0$ and $\mathcal{H}_{34}|\psi_{34}\rangle = 0$, it follows that

$$\langle\psi_{14}\psi_{23}|V_{13} + V_{14} + V_{23} + V_{24}|\psi_{12}\psi_{34}\rangle$$
$$= \langle\psi_{14}\psi_{23}| - \mathcal{H}_{12} - \mathcal{H}_{34} + V_{13} + V_{14} + V_{23} + V_{24}|\psi_{12}\psi_{34}\rangle$$
$$= \langle\psi_{14}\psi_{23}| - \mathcal{H}_{14} - \mathcal{H}_{23} + V_{12} + V_{13} + V_{42} + V_{43}|\psi_{12}\psi_{34}\rangle,$$

where we have used Eqs. (17) and (18) for the Hamiltonian of the system. Because $\mathcal{H}_{14}|\psi_{14}\rangle = 0$ and $\mathcal{H}_{23}|\psi_{23}\rangle = 0$, the above equation leads to

$$\langle\psi_{14}\psi_{23}|V_{13} + V_{14} + V_{23} + V_{24}|\psi_{12}\psi_{34}\rangle$$
$$= \langle\psi_{14}\psi_{23}|V_{12} + V_{13} + V_{42} + V_{43}|\psi_{12}\psi_{34}\rangle, \quad (27)$$

which is the relativistic generalization of the post-prior equivalence of the reaction matrix elements.

Just as in non-relativistic reaction theory [1], the equivalence is possible if and only if the interaction which induces the reaction is the same as the interaction that generates the composite bound states, and the reaction matrix element is evaluated using these bound state wave functions. The post-prior equivalence guarantees a unique result for the reaction cross section in the first-Born approximation.

4 Conclusion and Summary

The formulation of the relativistic two-body bound state problem in constraint dynamics allows a simple separation of center-of-mass and relative motion. The equation for relative motion can be cast in the form of a non-relativistic Schrödinger equation in the center-of-momentum system. The two-body bound state mass in the relativistic case is related to the eigenvalue of the Schrödinger equation by a simple algebraic equation, Eq. (13).

The relativistic constraint dynamics can be generalized for a many-particle system. Using the two-body solution as basis states for multi-particle dynamics, the N-body Hamiltonian can be separated into an unperturbed Hamiltonian and a residual interaction. The study of the dynamics involves the evaluation of the reaction matrix elements of the residual interaction using the wave functions of the composite particles.

In rearrangement reactions, because of the the freedom of partitioning the total Hamiltonian into different unperturbed and interaction parts, the evaluation of the reaction matrix elements can be carried out either in the post or prior forms. We show explicitly the equality of the post and prior reaction matrix elements for relativistic reaction of composite particles which guarantees a unique perturbation expansion in relativistic reaction of composite particles.

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